# Lagrange Interpolating Polynomial

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# Introduction

A census of the population of the India is taken every 10 years. The following table lists the population, in thousands of people, from 1951 to 2011.

| Year | 1951 | 1961 | 1971 | 1981 | 1991 | 2001 | 2011 |
|---|---|---|---|---|---|---|---|
| Population (in thousands) | 361,088 | 439,235 | 548,160 | 683,329 | 846,388 | 1,028,737 | 1,210,193 |

In reviewing these data, we might ask whether they could be used to provide a reasonable estimate of the population, say, in 1996, or even in the year 2014. Predictions of this type can be obtained by using a function that fits the given data.

This process is called **interpolation**.

# Weierstrass Approximation Theorem

One of the most useful and well-known classes of functions, mapping the set of real numbers into itself, is the class of **algebraic polynomials**, the set of functions of the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $n$ is a nonnegative integer, $a_0, a_1, \ldots, a_n$ are real constants, and $x$ is a variable.

Its OK for a polynomial to have more than one variable, but we will only talk about polynomials with 1 variable.

The individual pieces of a polynomial are called **terms**. The polynomial $5x^4 - 7x^3 + 2x - 11$ has 4 terms. They are $5x^4, 7x^3, 2x$ and 11.

The term $-11$ doesnt contain a variable. For this reason, its called the **constant term**. The degree of the polynomial is the largest exponent. Here, the degree is 4.

Given any function, defined and continuous on a closed and bounded interval, there exists a polynomial that is as "close" to the given function as desired. This result is expressed precisely in the following theorem.

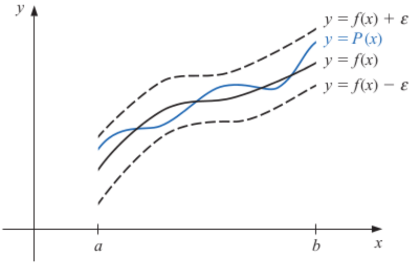
### Theorem (Weierstrass Approximation Theorem)

*Suppose that $f$ is defined and continuous on $[a, b]$. For each $\varepsilon > 0$, there exists a polynomial $P(x)$, with the property that*

$$|f(x) - P(x)| < \varepsilon, \text{ for all } x \in [a, b].$$

# Why polynomials are important?

Weierstrass Approximation Theorem is illustrated in the figure.




In science and engineering, polynomials arise everywhere.

An important reason for considering the class of polynomials in the approximation of functions is that the "derivative and indefinite integral of a polynomial" are easy to determine and they are also polynomials.

For these reasons, polynomials are often used for approximating continuous functions.

# Taylor polynomials are not useful for interpolation

The Taylor polynomials (discussed earlier) are one of the fundamental building blocks of numerical analysis.

The Taylor polynomials agree as closely as possible with a given function at a specific point, but they concentrate their accuracy near that point.

A good interpolation needs to provid a relatively accurate approximation over an entire interval, and Taylor polynomials do not generally do this.

## Example

*Taylor polynomials of various degree for* $f(x) = 1/x$ *about* $x_0 = 1$ *are*

$$P_n(x) = \sum_{k=0}^{n} (-1)^k (x-1)^k.$$

*When we approximate* $f(3) = 1/3$ *by* $P_n(3)$ *for larger values of* $n$*, the approximations become increasingly inaccurate, as shown in the following table.*

| n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $P_n(3)$ | 1 | -1 | 3 | -5 | 11 | -21 | 43 | -85 |

# Taylor polynomials are not appropriate for interpolation

Since the Taylor polynomials have the property that all the information used in the approximation is concentrated at the single point $x_0$, it is not uncommon for these polynomials to give inaccurate approximations as we move away from $x_0$. This limits Taylor polynomial approximation to the situation in which approximations are needed only at points close to $x_0$.

For ordinary computational purposes it is more efficient to use methods that include information at various points.

The primary use of Taylor polynomials in numerical analysis is not for approximation purposes, but for the derivation of numerical techniques and for error estimation.

Since the Taylor polynomials are not appropriate for interpolation, alternative methods are needed.

The problem of determining a polynomial of degree one that passes through the distinct points $(x_0, y_0)$ and $(x_1, y_1)$ is the same as approximating a function $f$ for which $f(x_0) = y_0$ and $f(x_1) = y_1$ by means of a first-degree polynomial interpolating, or agreeing with, the values of $f$ at the given points.

We first define the functions

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \text{ and } L_1(x) = \frac{x - x_0}{x_1 - x_0},$$

and then define

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1).$$

Since $L_0(x_0) = 1, L_0(x_1) = 0, L_1(x_0) = 0,$ and $L_1(x_1) = 1$, we have $P(x_0) = y_0$ and $P(x_1) = y_1$. So $P$ is the unique linear function passing $(x_0, y_0)$ and $(x_1, y_1)$.

# Lagrange Interpolating Polynomial

To gneralize the concept of linear interpolation, consider the construction of a polynomial of degree at most $n$ that passes through the $n+1$ points $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$.

In this case we need to construct, for each $k = 0, 1, 2, \ldots, n$, a function $L_k(x)$ (called **Lagrange basis**, also called the $n$th **Lagrange interpolating polynomial**) with the property that $L_k(x_i) = 0$ when $i \neq k$ and $L_k(x_k) = 1$, hence

$$L_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{(x - x_i)}{(x_k - x_i)}.$$

The interpolating polynomial is easily described once the form of $L_k$ is known, by the following theorem.

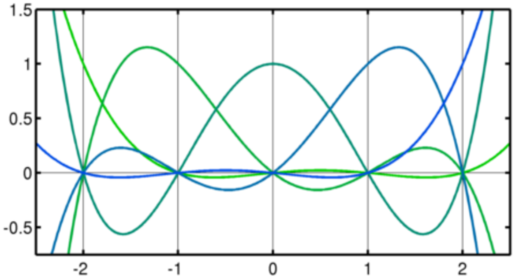
### Theorem

*If $n+1$ points $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$ are given, then a unique polynomial $P(x)$ of degree at most $n$ exists with $f(x_k) = P(x_k)$ for each $k = 0, 1, \ldots, n$. This polynomial is given by*

$$P(x) = \sum_{k=0}^{n} f(x_k) L_k(x).$$

# Graph of Lagrange Interpolating Polynomial

Given 5 points $(x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_4, f(x_4))$, a sketch of the graph of a typical $L_k$ is shown in Figure.



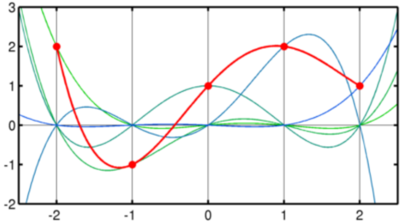

Note how each basis polynomial has a value of 1 for $x = x_k$ $(0 \leq k \leq 4)$, and a value of 0 at all other sample locations.

# Example

Simply multiplying each basis with the corresponding sample value, and adding them all up yields the interpolating polynomial

$$P(x) = \sum_{k=0}^{4} f(x_k) L_k(x).$$

The 5 weighted polynomials are $L_k(x)f(x_k)$ $(0 \leq k \leq 4)$ and their sum (red line) is the interpolating polynomial $P(x)$ (red line) which is shown in the following figure.



# How to calculate error bound?

The next step is to calculate a remainder term or bound for the error involved in approximating a function by an interpolating polynomial. This is done in the following theorem.

## Theorem (An Important Result for Error Formula)

*Suppose $x_0, x_1, \ldots, x_n$ are distinct numbers in the interval $[a, b]$ and $f \in C^{n+1}[a, b]$. Then, for each $x \in [a, b]$, a number $\xi(x)$ (generally unknown) in $(a, b)$ exists with*

$$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1)\cdots(x - x_n),$$

*where $P(x)$ is the interpolating polynomial given by $P(x) = \sum_{k=0}^{n} f(x_k)L_k(x)$.*

# Error Analysis

The error formula is an important theoretical result because Lagrange polynomials are used extensively for deriving numerical differentiation and integration methods.

Error bounds for these techniques are obtained from the “Lagrange error formula”.

Note that the error for the Lagrange polynomial is quite similar to that for the Taylor polynomial.

# Bound for the Error Involved

The $n$th Taylor polynomial about $x_0$ concentrates all the known information at $x_0$ and has an error term of the form

$$\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)^{n+1}.$$

The Lagrange polynomial of degree $n$ uses information at the distinct numbers $x_0, x_1, \ldots, x_n$ and, in place of $(x - x_0)^{n+1}$, its error formula uses a product of the $n + 1$ terms $(x - x_0)(x - x_1) \cdots (x - x_n)$:

$$\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n).$$

# References


- Richard L. Burden and J. Douglas Faires, "*Numerical Analysis – Theory ad Applications*", Cengage Learning, New Delhi, 2005.
- Kendall E. Atkinson, "*An Introduction to Numerical Analysis*", John Wiley & Sons, Delhi, 1989.